All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the definition of convergence of a sequence $\langle x_n \rangle$ to a number $\ell \in \mathbb{R}$.

   (b) Given two convergent sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ with $x_n \to x$ and $y_n \to y$, prove that the sequence $\langle x_n + y_n \rangle$ converges to $x + y$.

   (c) Using only the definition of convergence, prove that $\lim_{n\to\infty} \dfrac{2n+1}{3n-1} = \dfrac{2}{3}$.

   (d) Using any techniques at your disposal, compute the limit of each of the following sequences, or if the sequence diverges, explain why:

   $$\text{(i) } x_n = \frac{n^5 + 2n^3 + 6n}{2n^5 + 1}, \qquad \text{(ii) } x_n = 1 + (-1)^n - \frac{1}{n}.$$

2. (a) Given a real number L and a function f defined on an interval (a, b), state the definition of "$\lim_{x\to a^+} f(x) = L$".

   (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ -1/x & \text{if } 0 < x \le 1, \\ 2x - 3 & \text{if } x > 1. \end{cases}$$

   Prove carefully (using ϵ and δ) that $\lim_{x\to 1^+} f(x) = -1$.

   (c) For the function f in part (b), what is the set of all points in $\mathbb{R}$ at which f is continuous? Explain briefly. (For this part you may assume any results proved in the course.)

   (d) Determine whether the function f in part (b) is bounded on each of the intervals $[0, 2]$ and $[1/2, 2]$. Give brief explanations, quoting any theorems that you find helpful.

3. (a) For the set $S \subset \mathbb{R}$ defined as follows, find $\inf S$ and $\min S$ if they exist, and justify your answer:

   $$S = \{2^{-k} \mid k \in \mathbb{N}\}.$$

   (b) Prove that if $\langle x_n \rangle$ is a decreasing sequence that is bounded below, then it converges to the infimum of the set $\{x_n \mid n \in \mathbb{N}\}$.

   (c) Prove that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point, i.e. there exists a $c \in [0, 1]$ such that $f(c) = c$.

4. (a) Determine (with explanations) whether each of the following series converges absolutely, converges conditionally, or diverges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}+1}$ (ii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{n^2}+\frac{1}{\sqrt{n}}\right)$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!} \sin\left(n^{2013}\right)$

(b) Prove that if $|b_n| \leq a_n$ for all $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ converges absolutely.

5. (a) Use the ϵ-δ definition of the limit to prove the following version of the *sandwich theorem*: if f, g and h are functions on $\mathbb{R}$, $c \in \mathbb{R}$ is a number such that $f(x) \leq g(x) \leq h(x)$ for all $x > c$, and

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} h(x) = L,$$

then $\displaystyle\lim_{x \to c^+} g(x) = L$. State also (but do not prove) the corresponding theorem about limits as $x \to c^-$.

(b) Suppose f, g and h are functions on $\mathbb{R}$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$, and f and h are both differentiable at some point $c \in \mathbb{R}$, with

$$f(c) = h(c) = L \quad \text{and} \quad f'(c) = h'(c) = M.$$

Show that g is then also differentiable at c, and $g'(c) = M$.

(c) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = x^4 \sin(1/x)$ for $x \neq 0$ is differentiable at 0. What is $f'(0)$?

6. Throughout this problem, you may assume without proof all the familiar properties of the functions e^x and $\ln x$, e.g. $e^{x+y} = e^x e^y$, $e^{\ln x} = x$, $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = 1/x$.

(a) Show that $\displaystyle\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$. *[Hint: interpret the limit as a derivative.]*

(b) Suppose F is a function defined on the interval $(-1, 1)$, except possibly at 0, and suppose that $\displaystyle\lim_{x \to 0} F(x) = L$. Prove that if $\langle x_n \rangle$ is any sequence with $x_n \neq 0$ for all n and $x_n \to 0$, then $F(x_n) \to L$.

(c) Use the results of parts (a) and (b) to show that $\dfrac{\ln(1+1/n)}{1/n} \to 1$, and deduce that $\left(1+\dfrac{1}{n}\right)^n \to e$.

(d) Show that there exists a number $N > 0$ such that $\sqrt[n]{2} < 1 + \dfrac{1}{n}$ for all $n > N$. *[Hint: use the convergence from part (c) and the fact that $e > 2$.]*